Department for
Applied Mathematics


Sören Bartels, Philipp Tscherner

# Practical Exercises on Convexification

27 April 2023

---

**Exercise 1** (Rank-one convexification)**.** The one-dimensional convexification can be realized as follows: Let $(f_j)_{j=0,\dots,J}$ be a sequence of function values associated with grid points $x_j = x_0 + hj$. Set $g_j = f_j$ for $j = 0, \dots, J$. Then for $j = 2, \dots, J$ fix $g_j = f_j$ if

$$\frac{g_{j-1} - g_{j-2}}{x_{j-1} - x_{j-2}} \leq \frac{g_j - g_{j-1}}{x_j - x_{j-1}}.$$

Otherwise determine the smallest $k \in \mathbb{N}_+$ with $k \leq j - 2$ such that

$$\frac{g_{j-k} - g_{j-k-1}}{x_{j-k} - x_{j-k-1}} \leq \frac{g_j - g_{j-k}}{x_j - x_{j-k}}$$

and replace $g_{j-k+m}$ for $m = 1, \dots, k$ by

$$g_{j-k+m} = g_{j-k} + (x_{j-k+m} - x_{j-k}) \frac{g_j - g_{j-k}}{x_j - x_{j-k}}.$$

Implement the above algorithm and apply it to an appropriate set of function values. What assumptions does $f$ need to fulfill such that the described method results in a convex function $g$?

**Exercise 2** (Energy minimization)**.** For a uniform partition $\mathscr{T}_h$ of $\Omega = (0, 1)$ with meshsize $h = 1/N$ and

$$\mathscr{S}^1(\mathscr{T}_h) := \{v_h \in C(\overline{\Omega}) : v_h|_T \in P_1(T) \text{ for all } T \in \mathscr{T}_h\}$$

we seek a minimizer $u_h \in \mathscr{S}^1(\mathscr{T}_h)$ of the functional

$$I_h(u_h) = \frac{1}{4} \int_0^1 \left(|u_h'|^2 - 1\right)^2 \,\mathrm{d}x + \frac{1}{2} \int_0^1 u_h^2 \,\mathrm{d}x.$$

Implement a function energy that returns the value $I_h(u_h)$ of a function $u_h \in \mathscr{S}^1(\mathscr{T}_h)$. Realize the practical minimization of the energy for a random initial guess satisfying $u_h(0) = u_h(1) = 0$ and $u_h(x) \in [-h, h]$. Plot the numerical solutions in each step.
*Hint:* In OCTAVE the function fminserach(@energy,u) can be used.

**Exercise 3** (Polyconvexification)**.** (i) Download the script polyconvexification.m.
(ii) Create a subroutine grid_gen_mat that realizes the grid $\mathscr{N}_{\delta,r} = \delta\mathbb{Z}^{2\times 2} \cap K_r$, where $K_r := \{F \in \mathbb{R}^{2\times 2} : |F|_\infty \leq r\}$. How many elements $F \in \mathbb{R}^{2\times 2}$ does the set $\mathscr{N}_{\delta,r}$ contain?
(iii) Solve the minimization problem $W^{\mathrm{pc}}_{\mathscr{N}_{\mathrm{active}}}$ inside the function lin_prog defined via

$$W^{\mathrm{pc}}_{\mathscr{N}}(F) = \inf\Big\{ \sum_{A\in\mathscr{N}} \theta_A W(A) : \theta_A \geq 0,\ \sum_{A\in\mathscr{N}} \theta_A = 1,\ \sum_{A\in\mathscr{N}} \theta_A T(A) = T(F) \Big\}$$

to obtain the convex coefficients $(\theta_A)_{A\in\mathscr{N}}$ associated to the fixed nodes $A \in \mathscr{N} \subset \mathbb{R}^{2\times 2}$.
*Hint:* In OCTAVE the function glpk solves a linear minimization problem subject to equality and inequality constraints.